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A multiperiod two-echelon multicommodity capacitated plant location problem

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Abstract

In this paper we deal with a facility location problem where one desires to establish facilities at two different distribution levels by selecting the time periods. Our model intends to minimize the total cost for meeting demands for all the products specified over the planning horizon at various customer locations while satisfying the capacity requirements of the production plants and intermediate warehouses. We address this problem by means of a formulation as a mixed integer programming problem. A Lagrangean relaxation is proposed to solve the problem, together with a heuristic procedure that constructs feasible solutions of the original problem from the solutions at the lower bounds obtained by the relaxed problems. Computational tests are provided showing the good performance of this approach for a wide range of problems.

Keywords: Multiperiod multi-echelon facility-location; Integer programming; Lagrangean dual; Heuristic

1. Introduction

In many real world situations where large companies manufacture and distribute products it is necessary to locate production plants or warehouses to deliver goods or products to final customers in order to meet their demands. This is the case of health-care products, spare parts of cars or catalogues in travel agencies. If the admissible locations of these facilities are finite and known in advance, we cope with a discrete plant location problem. These problems have been widely studied and roughly speaking are classified into: (1) uncapacitated plant location problems (UPLP), and (2) capacitated plant location problems (CPLP). Both kinds of problems can be formulated as mixed integer programming problems (see [1]). Nevertheless, obtaining their exact solutions in polynomial time is not possible because Krarup and Pruzan [16] prove that even the UPLP belongs to the class of NP-hard problems. See e.g. the papers of Aikens [1], Drezner [8] or Daskin [7] for a good overview of these kinds of problems and their extensions.

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Among these extensions we shall focus on two of them. The first one consists of introducing the dynamic aspect into the problem. In this case, not only the transportation plan but also the time-staged establishment of the facilities are decision variables (see e.g. [21,22], or more recently [6]). The second one consists of the assumption of a certain structure in the transportation pattern. This is to say, the transportation follows a two-step path. These models have been hardly studied in the classical literature of location (see e.g. [15,18,20]), although recently a detailed analysis can be found in [7] or [17]. The main difference in these models is that the products are delivered from the production plants to the warehouses and then from the warehouses to the final customers (or retailers). Therefore, the decision problem consists of locating the plants and the warehouses and determining the amount of the different products that will be delivered from each open plant to each open warehouse and from each open warehouse to each final customer. Obviously, capacities may or may not be considered. That structure has been sometimes called two-echelon approach (see [3,5,17]).

Despite the generality of these models, the most natural framework for these problems is the combination of these two approaches. That is, the joint consideration of multiperiod and multi-echelon aspects (see Fig. 1). Nevertheless, as far as we know, this approach has never been addressed before and it can be considered as an introduction to a new location problem.

In this paper, we deal with a multiperiod two-echelon multicommodity capacitated location problem. We assume that the capacities of plants and warehouses, as well as demands and transportation costs change over T time periods. We do not consider holding decisions. Our goal is to minimize the total cost for meeting demands for the different products specified over time at various customer locations. Although no-real application has motivated this model, it perfectly applies to those situations where intermediate distribution and seasonal demand exist. The formulation permits both the opening of new facilities and the closing of existing ones. This is a very large mixed integer programming problem. For instance, for a problem with 50 customers, 20 warehouses, 20 plants, two different products and four time periods we shall have 11 360 variables and 764 constraints. Our computational experience shows that solving exactly this problem by branch and bound needs prohibitive CPU time (see Table 3). Therefore, we propose an alternative approach. We present a Lagrangean relaxation scheme incorporating a dual ascent method together with a heuristic construction phase method which shows in computational test to provide good feasible solutions for this problem (see [2,4,9,11,12,14] for similar analysis in different problems). As commented above, no comparative testing with other procedures in the literature can be reported because as far as we know, this is the first time that this problem has been addressed.

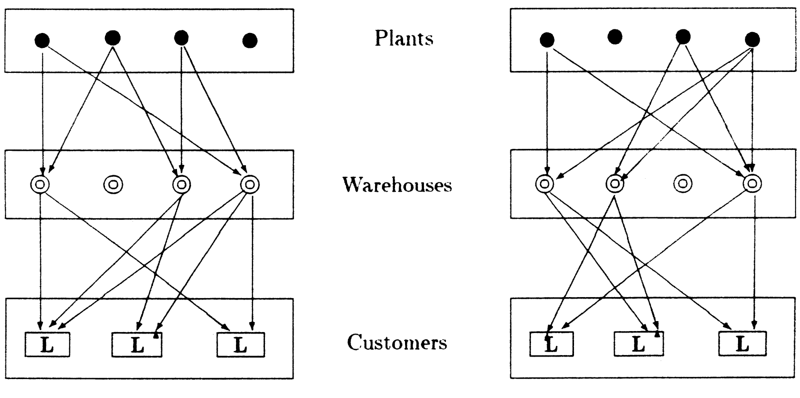

Fig. 1. Time period t. Time period $t + 1$.

The paper is organized as follows. In Section 2 the mathematical formulation of the model is presented, together with a suitable reformulation more convenient for our optimization purposes. In Section 3 a Lagrangean relaxation is proposed for this problem and its solution is presented. Section 4 develops the heuristic phase of our solution method. Section 5 is devoted to the computational results and Section 6 to the conclusions. The paper ends with an Appendix A with some technical results.

2. The model

The multiperiod two-echelon multicommodity capacitated plant location problem we deal with has the objective of minimizing the total cost for meeting demands of the different products specified over time at various customer locations. The version of the problem that we consider in the paper assumes the following hypotheses. There are not holding decisions. The sets of customers and products, together with the feasible locations for the facilities (plants and warehouses) are considered fixed and known beforehand. Therefore, they do not change over the time horizon. It is usual to consider seasons or months as a typical period length for this kind of problem. Then, the time horizon is chosen in accordance with the period lengths and the planning horizon. In addition, we will denote by:

$I = \{1, \dots, n\}$ set of customers, indexed by $i \in I$.
$L = \{1, \dots, q\}$ set of product types, indexed by $l \in L$.
$J = \{1, \dots, m\}$ set of possible location for warehouses, indexed by $j \in J$.
$K = \{1, \dots, p\}$ set of possible location for plants, indexed by $k \in K$.

At the beginning of the first time period there exists a subset K_c (respectively J_c) of the whole set of feasible locations for the plants (respectively warehouses) where operating facilities are established. These facilities can be closed at the end of any time period $t \in \{1, \dots, T\}$, but once closed they cannot be re-opened. We denote by K_0 (respectively J_0) the set of feasible locations where there does not exist open plants (respectively warehouses). These facilities can be opened at the beginning of any time period and it is also assumed that if they were open they would not be closed. This hypothesis is quite reasonable. In real-life applications the opening/closing of final retailers usually leads to a loss of market because customers require a certain regularity to patronize a particular facility. In addition, this phenomenon increases the operating cost.

Additionally, we assume that a minimum number of plants and warehouses must be open at the first and last time period which assures a minimum coverage of the demand at the beginning and after the time horizon. Let us denote by ND^1, ND^T (respectively NC^1, NC^T) the minimum number of warehouses (respectively plants) open at the beginning of the first time period and at the end of the last time period.

We consider the situation where both plants and warehouses have limited capacity which depends on the time period. We denote by:

$W_j^t :=$ capacity of warehouse j at time period t.
$C_k^t :=$ capacity of plant k at time period t.
$d_{il}^t :=$ demand of product l at customer i during time period t.

Finally, we assume a cost structure that includes both transportation costs of goods and maintenance costs. For the elements of this problem we will use the following notation:

$c_{ijl}^t :=$ transportation cost per unit of product l from warehouse j to customer i at time period t.
$b_{jkl}^t :=$ transportation cost per unit of product l from plant k to warehouse j at time period t.
$f_j^t :=$ operating cost of a warehouse open at position j during time period t.
$g_k^t :=$ operating cost of a plant open at position k during time period t.

Notice that we do not explicitly have an installation or setup cost for the facilities. This is because the facilities which belong to the set J_c (respectively K_c) are already open at the beginning of the first period. Therefore, they do not have installation of setup cost. On the second hand, we consider that the facilities which belong to J_0 (respectively K_0) have a fixed setup cost which is charged at period T. This is possible because, once these facilities are opened they will never be closed until the end of the planning horizon. The decision variables of the problem are:

x^t_{ijl} := fraction (regarding to d^t_{il}) of product l delivered to customer i from warehouse j at time period t.
y^t_{jkl} := fraction (regarding to W^t_j) of product l sent to warehouse j from plant k at time period t.

$$u^t_j = \begin{cases} 1 & \text{if warehouse } j \text{ is open at the beginning of time period } t, \\ 0 & \text{otherwise.} \end{cases}$$

$$v^t_k = \begin{cases} 1 & \text{if plant } k \text{ is open at the beginning of time period } t, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we denote by $v(\cdot)$ the optimal objective value of Problem $(\cdot)$. Using these conventions, the mathematical formulation of this problem is

$$(P) \quad \min g(x,y,u,v) := \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{q} c^t_{ijl}x^t_{ijl}d^t_{il} + \sum_{t=1}^{T}\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} b^t_{jkl}y^t_{jkl}W^t_j + \sum_{t=1}^{T}\sum_{j=1}^{m} f^t_j u^t_j + \sum_{t=1}^{T}\sum_{k=1}^{p} g^t_k v^t_k$$

s.t.

$$\sum_{j=1}^{m} x^t_{ijl} \geqslant 1 \quad \forall i,\ \forall l,\ \forall t, \tag{1}$$

$$\sum_{i=1}^{n}\sum_{l=1}^{q} d^t_{il}x^t_{ijl} \leqslant W^t_j u^t_j \quad \forall j,\ \forall t, \tag{2}$$

$$\sum_{k=1}^{p} W^t_j y^t_{jkl} \geqslant \sum_{i=1}^{n} d^t_{il}x^t_{ijl} \quad \forall j,\ \forall l,\ \forall t, \tag{3}$$

$$\sum_{j=1}^{m}\sum_{l=1}^{q} W^t_j y^t_{jkl} \leqslant C^t_k v^t_k \quad \forall k,\ \forall t, \tag{4}$$

$$\sum_{j=1}^{m} u^1_j \geqslant ND^1, \quad \sum_{j=1}^{m} u^T_j \geqslant ND^T, \tag{5}$$

$$\sum_{k=1}^{p} v^1_k \geqslant NC^1, \quad \sum_{k=1}^{p} v^T_k \geqslant NC^T, \tag{6}$$

$$u^1_j = 1 \quad \forall j \in J_c; \quad u^t_j \geqslant u^{t+1}_j \quad \forall j \in J_c \forall t; \quad u^t_j \leqslant u^{t+1}_j \quad \forall j \in J_0, \quad \forall t, \tag{7}$$

$$v^1_k = 1 \quad \forall k \in K_c; \quad v^t_k \geqslant v^{t+1}_k \quad \forall k \in K_c\ \forall t; \quad v^t_k \leqslant v^{t+1}_k \quad \forall k \in K_0, \quad \forall t, \tag{8}$$

$$x_{ijl}^{t}, y_{jkl}^{t} \geqslant 0 \quad \forall i,\ \forall j,\ \forall k,\ \forall l,\ \forall t, \tag{9}$$

$$u_j^t, v_k^t \in \{0,1\} \quad \forall j,\ \forall k,\ \forall t. \tag{10}$$

The constraints (1) and (3) refer to the demand. The constraints (1) guarantee meeting the demand of each customer for each one of the products in each time period t. Notice that warehouse j needs an amount of product l equal to the sum of the amounts of this product that it delivers to the final customers. The constraints (2) and (4) refer to capacity. The constraint (2) assures that the total number of units delivered from warehouse j is not greater than its capacity in each time period t. The constraint (4) is similar to (2) but focused towards plants rather that warehouses. The constraints (5) and (6) state the minimum number of warehouses and plants that must be open at the first and last time period. The constraints (7) and (8) describe the sets $J = J_0 \cup J_c$ and $K = K_0 \cup K_c$.

This is a formulation often used for multiperiod models (see e.g. [19] or [6]). Thus, this formulation simplifies the understanding proccess of our model to the readers accustomed to read papers in this field. Nevertheless, from the point of view of our resolution approach, it is more convenient to deal with an alternative formulation which will be proven equivalent (see Theorem 1). To this end, the following variables z_j^t and ζ_k^t are introduced:

$$\forall j \in J_0,\ \forall t, \quad z_j^t = \begin{cases} 1 & \text{if a warehouse is established at } j \text{ at the beginning of time period } t, \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{matrix}\forall j \in J_c, \\ \forall t < T-1,\end{matrix} \quad z_j^t = \begin{cases} 1 & \text{if existing warehouse at } j \text{ is removed at the end of time period } t, \\ 0 & \text{otherwise.} \end{cases}$$

$$\forall j \in J_c \quad z_j^T = \begin{cases} 1 & \text{if warehouse } j \text{ is open during all the planning horizon,} \\ 0 & \text{otherwise.} \end{cases}$$

ζ_k^t is analogously defined for the set of plants.

And the costs are defined as follows:

$$F_j^t = \sum_{r=t}^{T} f_j^r, \quad \text{total cost of warehouse } j \text{ being established in time period } t \quad \forall t,\ \forall j \in J_0.$$

$$F_j^t = \sum_{r=1}^{t} f_j^r, \quad \text{total cost of warehouse } j \text{ removed at the end of time period } t \quad \forall t,\ \forall j \in J_c.$$

$$G_k^t = \sum_{r=t}^{T} g_k^r, \quad \text{total cost of plant } k \text{ being established in time period } t \quad \forall t,\ \forall k \in K_0.$$

$$G_k^t = \sum_{r=1}^{t} g_k^r, \quad \text{total cost of plant } k \text{ removed at the end of time period } t \quad \forall t,\ \forall k \in K_c.$$

Let

$$T_{jt} = \begin{cases} \{1,\ldots,t\} & \text{if } j \in J_0 \\ \{t,\ldots,T\} & \text{if } j \in J_c \end{cases} \quad \text{and} \quad T_{kt} = \begin{cases} \{1,\ldots,t\} & \text{if } k \in K_0 \\ \{t,\ldots,T\} & \text{if } k \in K_c \end{cases}.$$

Problem (P) can be reformulated in terms of the variables z_j^t, ζ_k^t and the costs F_j^t, G_k^t, namely (P'):

$$(P') \quad \min f(x,y,z,\zeta) := \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{q} c_{ijl}^t x_{ijl}^t d_{il}^t + \sum_{t=1}^{T}\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} b_{jkl}^t y_{jkl}^t W_j^t + \sum_{t=1}^{T}\sum_{j=1}^{m} F_j^t z_j^t + \sum_{t=1}^{T}\sum_{k=1}^{p} G_k^t \zeta_k^t$$

s.t.

$$\sum_{j=1}^{m} x_{ijl}^t \geqslant 1 \quad \forall i,\ \forall l,\ \forall t, \tag{1}$$

$$\sum_{i=1}^{n}\sum_{l=1}^{q} d_{il}^t x_{ijl}^t \leqslant W_j^t \sum_{r\in T_{jt}} z_j^r \quad \forall j,\ \forall t, \tag{2a}$$

$$\sum_{k=1}^{p} W_j^t y_{jkl}^t \geqslant \sum_{i=1}^{n} d_{il}^t x_{ijl}^t \quad \forall j,\ \forall l,\ \forall t, \tag{3}$$

$$\sum_{j=1}^{m}\sum_{l=1}^{q} W_j^t y_{jkl}^t \leqslant C_k^t \sum_{r\in T_{kt}} \zeta_k^r \quad \forall k,\ \forall t, \tag{4a}$$

$$\sum_{j\in J0} z_j^1 + \sum_{j\in Jc}\sum_{t=1}^{T} z_j^t \geqslant ND^1, \qquad \sum_{j\in J0}\sum_{t=1}^{T} z_j^t + \sum_{j\in Jc} z_j^T \geqslant ND^T, \tag{5a}$$

$$\sum_{k\in K0} \zeta_k^1 + \sum_{k\in Kc}\sum_{t=1}^{T} \zeta_k^t \geqslant NC^1, \qquad \sum_{k\in Ko}\sum_{t=1}^{T} \zeta_k^t + \sum_{k\in Kc} \zeta_k^T \geqslant NC^T, \tag{6a}$$

$$\sum_{t=1}^{T} z_j^t = 1 \quad \forall j \in J_c; \qquad \sum_{t=1}^{T} z_j^t \leqslant 1 \quad \forall j \in J_0, \tag{7a}$$

$$\sum_{t=1}^{T} \zeta_k^t = 1 \quad \forall k \in K_c; \qquad \sum_{t=1}^{T} \zeta_k^t \leqslant 1 \quad \forall k \in K_0, \tag{8a}$$

$$x_{ijl}^t, y_{jkl}^t \geqslant 0 \quad \forall i,\ \forall j,\ \forall k,\ \forall l,\ \forall t, \tag{9}$$

$$z_j^t, \zeta_k^t \in \{0,1\} \quad \forall j,\ \forall k,\ \forall t. \tag{10a}$$

Theorem 1. *If (x,y,u,v) is a feasible solution of problem (P) then there exists a feasible solution (x',y',z,ζ) of problem (P') such that $g(x,y,u,v) = f(x',y',z,\zeta)$. Conversely, if (x',y',z,ζ) is a feasible solution of problem (P') then there exists a feasible solution (x,y,u,v) of problem (P) such that $f(x',y',z,\zeta) = g(x,y,u,v)$.*

The proof is included in Appendix A.

Theorem 1 proves that both formulations are equivalent in the sense that they provide the same set of optimal solutions. From now on, we will always deal with problem (P').

Problem (P') is a mixed-integer programming problem which includes as a particular instance the UPLP. Since our problem includes as a particular case the UPLP and this problem is NP-hard [16] one cannot expect to solve exactly large sizes of problem (P') in polynomial time. For this reason, we will adopt a heuristic method to solve (P') for those instances. It is based on: (1) using a Lagrangean relaxation, and (2) using an "ad hoc" procedure obtaining a feasible solution from the solutions of the relaxed problems.

3. Decomposition of the problem: Lagrangean relaxation

In this section, we consider a relaxation of problem (P') obtained relaxing the constraints that ensure that demands are met. To this end, we associate nonnegative multipliers $\mu_{il}^t \geqslant 0$ to the constraints (1) and $\lambda_{jl}^t \geqslant 0$ to the constraints (3). Therefore the relaxed problem $LR(\lambda, \mu)$ is:

$$\begin{aligned} LR(\lambda,\mu) \quad \min f_{\lambda,\mu}(x,y,z,\zeta) := {} & \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{q} c_{ijl}^t x_{ijl}^t d_{il}^t + \sum_{t=1}^{T}\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} b_{jkl}^t y_{jkl}^t W_j^t + \sum_{t=1}^{T}\sum_{j=1}^{m} F_j^t z_j^t \\ & + \sum_{t=1}^{T}\sum_{k=1}^{p} G_k^t \zeta_k^t + \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{l=1}^{q} \mu_{il}^t \left(1 - \sum_{j=1}^{m} x_{ijl}^t\right) \\ & + \sum_{t=1}^{T}\sum_{j=1}^{m}\sum_{l=1}^{q} \lambda_{jl}^t \left(\sum_{i=1}^{n} d_{il}^t x_{ijl}^t - \sum_{k=1}^{p} W_j^t y_{jkl}^t\right) \end{aligned}$$

$$\text{s.t.} \quad (2a), (4a), (5a), (6a), (7), (8a), (9), (10a).$$

A little thought about problem $LR(\lambda, \mu)$ leads us to separate it into two subproblems, $LR1(\lambda, \mu)$ and $LR2(\lambda, \mu)$. These two problems are the following:

$$LR1(\lambda,\mu) \quad \min \quad \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{q}\left(c_{ijl}^t d_{il}^t + \lambda_{jl}^t d_{il}^t - \mu_{il}^t\right) x_{ijl}^t + \sum_{t=1}^{T}\sum_{j=1}^{m} F_j^t z_j^t$$

$$\text{s.t.} \quad (2a), (5a), (7a), x_{ijl}^t \geqslant 0, \quad z_j^t \in \{0,1\}$$

and

$$LR2(\lambda,\mu) \quad \min \quad \sum_{t=1}^{T}\sum_{j=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q}\left(b_{jkl}^t W_j^t - \lambda_{jl}^t W_j^t\right) y_{jkl}^t + \sum_{t=1}^{T}\sum_{k=1}^{p} G_k^t \zeta_k^t$$

$$\text{s.t.} \quad (4a), (6a), (8a), y_{jkl}^t \geqslant 0, \quad \zeta_k^t \in \{0,1\}.$$

These problems can be solved independently and their solutions can be used to solve $LR(\lambda, \mu)$. Once the problems $LR1(\lambda, \mu)$ and $LR2(\lambda, \mu)$ have been solved, the value of $LR(\lambda, \mu)$ is given by the following proposition whose proof is obvious.

Proposition 1.

$$v(LR(\lambda,\mu)) = v(LR1(\lambda,\mu)) + v(LR2(\lambda,\mu)) + \sum_{t=1}^{T}\sum_{i=1}^{n}\sum_{l=1}^{q} \mu_{il}^t.$$

3.1. Analysis of $LR1(\lambda, \mu)$

First of all, in order to solve $LR1(\lambda, \mu)$ we will leave constraints (5a) aside. Then, $LR1(\lambda, \mu)$ can be separated into m subproblems and once the solution of each subproblem is obtained, we will obligate constraints (5a) to be fulfilled in such a way that the optimal value for $LR1(\lambda, \mu)$ is obtained. This last step is justified in Proposition 3.

Provided that (5a) is removed $LR1(\lambda, \mu)$ can be separated into the following m subproblems, one for each $j = 1, \ldots, m$:

$$LR1_j(\lambda, \mu) \quad \min \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{l=1}^{q} \left(c_{ijl}^t d_{il}^t + \lambda_{jl}^t d_{il}^t - \mu_{il}^t \right) x_{ijl}^t + \sum_{t=1}^{T} F_j^t z_j^t$$

s.t.

$$\sum_{i=1}^{n} \sum_{l=1}^{q} d_{il}^t x_{ijl}^t \leqslant W_j^t \sum_{r \in T_{jt}} z_j^r \quad \forall t,$$

$$\sum_{t=1}^{T} z_j^t \leqslant 1 \text{ if } j \in J_0 \quad \text{or} \quad \sum_{t=1}^{T} z_j^t = 1 \text{ if } j \in J_c,$$

$$x_{ijl}^t \geqslant 0 \quad \forall i,\ \forall l,\ \forall t,$$

$$z_j^t \in \{0, 1\} \quad \forall t.$$

These problems are associated to each warehouse $j \in J$ and we can solve them independently.

In order to solve $LR1_j(\lambda, \mu)$, we distinguish two cases depending on either $j \in J_0$ or $j \in J_c$ because of the relationships that hold among the z_j^t variables on each case.

1. Let us assume $j \in J_0$. For each $t_0 = 1, \ldots, T$ let $LR1_{jt_0}(\lambda, \mu)$ be the following problem:

$$LR1_{jt_0}(\lambda, \mu) \quad \sum_{t=t_0}^{T} \sum_{i=1}^{n} \sum_{l=1}^{q} \left(c_{ijl}^t d_{il}^t + \lambda_{jl}^t d_{il}^t - \mu_{il}^t \right) x_{ijl}^t + F_j^{t_0}$$

s.t.

$$\sum_{i=1}^{n} \sum_{l=1}^{q} d_{il}^t x_{ijl}^t \leqslant W_j^t \quad \forall t \geqslant t_0,$$

$$x_{ijl}^t \geqslant 0 \quad \forall i,\ \forall l,\ \forall t \geqslant t_0.$$

2. Let us assume $j \in J_c$. For each $t_0 = 1, \ldots, T$ let $LR1_{jt_0}(\lambda, \mu)$ be the following problem:

$$LR1_{jt_0}(\lambda, \mu) \quad \min \sum_{t=1}^{t_0} \sum_{i=1}^{n} \sum_{l=1}^{q} \left(c_{ijl}^t d_{il}^t + \lambda_{jl}^t d_{il}^t - \mu_{il}^t \right) x_{ijl}^t + F_j^{t_0}$$

s.t.

$$\sum_{i=1}^{n} \sum_{l=1}^{q} d_{il}^t x_{ijl}^t \leqslant W_j^t \quad \forall t \leqslant t_0,$$

$$x_{ijl}^t \geqslant 0 \quad \forall i,\ \forall l,\ \forall t \leqslant t_0.$$

Proposition 2.
1. *If* $j \in J_0$, $\quad v(LR1_j(\lambda,\mu)) = \min\left\{ \min_{1 \leqslant t_0 \leqslant T} v(LR1_{jt_0}(\lambda,\mu)), 0 \right\}.$
2. *If* $j \in J_c$, $\quad v(LR1_j(\lambda,\mu)) = \min_{1 \leqslant t_0 \leqslant T} v(LR1_{jt_0}(\lambda,\mu)).$

Proof.
1. If $j \in J_0$, it may occur that: (a) for some $t_0 = 1,\ldots,T$, $z_j^{t_0} = 1$ and $z_j^t = 0$ for all $t \neq t_0$ (this assumption corresponds to the hypothesis that warehouse j is open at time period t_0), or (b) $z_j^t = 0$ for all t (this assumption corresponds to the hypothesis that warehouse j is never open).
   In case (a) we have, $z_j^{t_0} = 1$, $z_j^t = 0$ for all $t \neq t_0$. Then

$$\sum_{t=1}^{T}\sum_{j=1}^{m} F_j^t z_j = F_j^{t_0} \quad \text{and} \quad \sum_{r \in T_{jt}} z_j^r = \begin{cases} 1 & \text{if } t \geqslant t_0, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, $x_{ijl}^t \geqslant 0\ \forall i,\ \forall l,\ \forall t \geqslant t_0$ and $x_{ijl}^t = 0\ \forall i,\ \forall l,\ \forall t < t_0$. Applying these transformations to problem $LR1_j(\lambda,\mu)$ it becomes problem $LR1_{jt_0}(\lambda,\mu)$.

In case (b) $z_j^t = 0$ for all t. Then, the objective function of problem $LR1_j(\lambda,\mu)$ is zero. Thus, $v(LR1_j(\lambda,\mu))$ is the minimum between all these possibilities:

$$v(LR1_j(\lambda,\mu)) = \min\left\{ \min_{1 \leqslant t_0 \leqslant T} v(LR1_{jt_0}(\lambda,\mu)), 0 \right\}.$$

2. If $j \in J_c$, we have for some $t_0 = 1,\ldots,T$, $z_j^{t_0} = 1$ and $z_j^t = 0$ for all $t \neq t_0$ (this assumption corresponds to the hypothesis that warehouse j is closed at the end of t_0).
   Assume $z_j^{t_0} = 1$, $z_j^t = 0$ for all $t \neq t_0$. Then

$$\sum_{t=1}^{T}\sum_{j=1}^{m} F_j^t z_j = F_j^{t_0} \quad \text{and} \quad \sum_{r \in T_{jt}} z_j^r = \begin{cases} 1 & \text{if } t \leqslant t_0, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, $x_{ijl}^t \geqslant 0\ \forall i,\ \forall l,\ \forall t \leqslant t_0$ and $x_{ijl}^t = 0\ \forall i,\ \forall l,\ \forall t > t_0$. Applying these transformations to problem $LR1_j(\lambda,\mu)$ it becomes problem $LR1_{jt_0}(\lambda,\mu)$.

Thus, $v(LR1_j(\lambda,\mu))$ is the minimum between these possibilities:

$$v(LR1_j(\lambda,\mu)) = \min_{1 \leqslant t_0 \leqslant T} v(LR1_{jt_0}(\lambda,\mu)).$$

Notice that to solve $LR1_j(\lambda,\mu)$ we have only needed to solve T independent, continuous linear programming problems, $LR1_{jt_0}(\lambda,\mu)$. In addition, solving $LR1_j(\lambda,\mu)$ for each $j = 1,\ldots,m$ we obtain the time period in which the warehouse j has to be opened (if this is the case) or closed and therefore, the value of the integer variables z_j^t for all j,t. Let J^* be the set of indexes of those warehouses which belong to J_0 and have never been opened after this process.

In order to solve $LR1(\lambda,\mu)$, we obligate the constraints (5a) to be fulfilled. We denote by

$$\Delta 1(j) := \begin{cases} v(LR1_{j1}(\lambda,\mu)) - v(LR1_j(\lambda,\mu)), & \text{if } j \in J_0 \text{ and } z_j^1 = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

First of all, we check the number of warehouses open at $t = 1$. If this number is less than ND^1 we calculate $\Delta 1(j)$ for each $j \in J_0$ closed at $t = 1$ and we open, one at a time, those warehouses $j \in J_0$ with the smallest increment $\Delta 1(j)$ until the constraint is fulfilled. Let J^1 denote the set of warehouses open in this way at $t = 1$.

Once, we have done that for $t = 1$ we proceed in the same way with the time period T. We denote by

$$\Delta T(j) := \begin{cases} \min_{t_0} v(LR1_{jt_0}(\lambda,\mu)) & \text{if } j \in J^* \setminus J^1, \\ v(LR1_{jT}(\lambda,\mu)) - v(LR1_j(\lambda,\mu)) & \text{if } j \in J_c \text{ and } z_j^T = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

We check the number of warehouses open at T. If this number is less than ND^T, we compute $\Delta T(j)$ for each $j \in J^* \setminus J^1$ and for each $j \in J_c$ closed at T. We select, one at a time, those warehouses with the smallest increment $\Delta T(j)$ until the constraint is fulfilled. Let J^T denote the set of warehouses chosen in this way.

If there exists a warehouse $j \in J^* \cap J^T$ such that, $\min_{t_0} v(LR1_{jt_0}(\lambda,\mu)) = v(LR1_{j1}(\lambda,\mu))$ then, we set $J^1 = J^1 \setminus \{j_0\}$ where j_0 is such that $\Delta 1(j_0) = \max_{j \in J^1 \setminus J^*} \Delta 1(j)$.

The following result proves that this procedure gives us an optimal solution of $LR1(\lambda,\mu)$.

Proposition 3.

$$v(LR1(\lambda,\mu)) = \sum_{j=1}^{m} v(LR1_j(\lambda,\mu)) + \sum_{j \in J^1} \Delta 1(j) + \sum_{j \in J^T} \Delta T(j).$$

Proof. Once problem $LR1_j(\lambda,\mu)$ has been solved for each $j = 1,\ldots,m$, four different cases may occur:

1. The constraints (5a) are fulfilled. In this case $J^1 = \emptyset, J^T = \emptyset$. Therefore, it is obvious by the decomposition of the problem into the m independent subproblems $LR1_j(\lambda,\mu)$, that

$$v(LR1(\lambda,\mu)) = \sum_{j=1}^{m} v(LR1_j(\lambda,\mu)).$$

2. The number of warehouses open at $t = 1$ is less than ND^1 and the number of warehouses open at T is greater than or equal to ND^T. In this case, $J^T = \emptyset$ and the optimal solution of $LR1_j(\lambda,\mu)$ is obtained by opening at $t = 1$ the number of required warehouses with the smallest increment in the objective function. This increment is given by $\Delta 1(j)$. Therefore,

$$v(LR1(\lambda,\mu)) = \sum_{j=1}^{m} v(LR1_j(\lambda,\mu)) + \sum_{j \in J^1} \Delta 1(j).$$

3. The number of warehouses open at $t = 1$ is greater than or equal to ND^1 and the number of warehouses open at T is less than ND^T. In this case, $J^1 = \emptyset$ and we have to fulfill the requirement on ND^T. This can be done with those warehouses belonging to J^* and J_c closed before T.

If $j \in J^*$ which means that this warehouse is never open, we have $v(LR1_j(\lambda,\mu)) = 0$ and $v(LR1_{jt_0}(\lambda,\mu)) \geqslant 0$ for all $t_0 = 1,\ldots,T$. Then, if the warehouse j would have been opened, the smallest increment for the objective function would have been given by $\Delta T(j)$. On the other hand, let's assume that $j \in J_c$ was closed before the time period T. If it would not have been closed, the minimum increment for the objective function would have been $\Delta T(j)$. Then

$$v(LR1(\lambda,\mu)) = \sum_{j=1}^{m} v(LR1_j(\lambda,\mu)) + \sum_{j \in J^T} \Delta T(j).$$

4. The number of warehouses open at $t = 1$ (respectively at T) is less than ND^1 (respectively ND^T). We have to fulfill the constraints on ND^1 (respectively ND^T) in such a way that the increment of the objective function is minimum.

We start by opening the warehouses j with minimum $\Delta 1(j)$ at $t = 1$ until the constraint on ND^1 is fulfilled. For those $j \in J^* \cap J^1$, this implies that they are open at T as well. Once the constraint on ND^1 is fulfilled, we proceed in a similar way, but instead, using $\Delta T(j)$ to fulfill the constraint on ND^T. This process produces the smallest increment in the objective function in such a way that the constraints on ND^1, ND^T are fulfilled (except in the case that $j \in J^* \cap J^T$ exists).

In the case of $j \in J^* \cap J^T$, to fulfill the constraint on ND^T we choose a warehouse $j \in J^*$ and the minimum increment is given by opening it at $t = 1$. In this situation we can close one warehouse which belongs to $J^1 \setminus J^*$ reducing the objective function and the constraint on ND^1 is still fulfilled. The maximal reduction is given by that $j \in J^1 \setminus J^*$ with the maximum $\Delta 1(j)$ value. Using this policy we obtain at the end an optimal solution of $LR1(\lambda, \mu)$ and its value is

$$v(LR1(\lambda,\mu)) = \sum_{j=1}^{m} v(LR1_j(\lambda,\mu)) + \sum_{j \in J^1} \Delta 1(j) + \sum_{j \in J^T} \Delta T(j).$$

3.2. Analysis of $LR2(\lambda, \mu)$

In order to solve $LR2(\lambda, \mu)$, we use the same strategy as for $LR1(\lambda, \mu)$. Once (6a) is removed, we separate $LR2(\lambda, \mu)$, into p subproblems $LR2_k(\lambda, \mu)$, one for each $k = 1, \ldots, p$. Thus, we solve the problem for each plant. These p subproblems will be separated into T subproblems, $LR2_{kt_0}(\lambda, \mu)$ for each $t_0 \in \{1, \ldots, T\}$. Then we solve each one of them as we did for $LR1(\lambda, \mu)$.

Proposition 4.
1. $v(LR2_k(\lambda,\mu)) = \min\left\{ \min_{1 \leqslant t_0 \leqslant T} v(LR2_{kt_0}(\lambda,\mu)), 0 \right\} \quad \forall k \in K_0,$
2. $v(LR2_k(\lambda,\mu)) = \min_{1 \leqslant t_0 \leqslant T} v(LR2_{kt_0}(\lambda,\mu)) \quad \forall k \in K_c,$
3. $v(LR2(\lambda,\mu)) = \sum_{k=1}^{p} v(LR2_k(\lambda,\mu)) + \sum_{k \in K^1} \Delta' 1(k) + \sum_{k \in K^T} \Delta' T(k)$, *where* K^1, $K^T, \Delta' 1(k)$ *and* $\Delta' T(k)$ *are defined in the same way as* $J^1, J^T, \Delta 1(j)$ *and* $\Delta T(j)$, *respectively.*

The proof is similar to the proof of Propositions 2 and 3, once one subtitutes warehouses by plants. Therefore, the proof is left out.

The solution of $LR(\lambda, \mu)$ for each set of multipliers verifies the following well-known relation [10]:

$$v(P) \geqslant v(DL) := \max_{\lambda,\mu} v(LR(\lambda,\mu)).$$

Since $v(LR(\lambda, \mu))$ is a piecewise linear, concave function, we can use a subgradient approximation scheme to get the maximum or at least a good lower bound. Nevertheless, it may happen that this solution is not feasible (i.e., it does not verify the relaxed constraints) for problem (P'). Therefore, we approximate (DL) by several choices of multipliers and, using the better solution, we construct a feasible solution by means of a heuristic approach.

3.3. The subgradient method

For each pair of fixed multipliers λ, μ the function $v(LR(\lambda, \mu))$ is a piecewise linear, concave function because it can be written as a pointwise infimum of affine-linear functions. Therefore, we can obtain the subdifferential set of $v(LR(\cdot, \cdot))$ at any point.

Let $X^*(\lambda, \mu)$ be the whole set of extreme optimal solutions of $LR(\lambda, \mu)$, and denote by $e(\lambda, \mu)$ any of its elements. That is to say, $e(\lambda, \mu) = (x, y, z, \zeta) \in X^*(\lambda, \mu)$. Thus, we can write

$$v(LR(\lambda, \mu)) = f_{\lambda,\mu}(e(\lambda, \mu)) \quad \text{for any } e(\lambda, \mu) \in X^*(\lambda, \mu),$$

where $f_{\lambda,\mu}$ was already defined as the objective function of problem $LR(\lambda, \mu)$.

Then, a subgradient of the function $f_{\lambda,\mu}$ at λ, μ is given by

$$\partial f_{\lambda,\mu}(e(\lambda, \mu)) = \begin{bmatrix} \sum_i d_{il}^t x_{ijl}^t - \sum_k W_j^t y_{jkl}^t \\ 1 - \sum_j x_{ijl}^t \end{bmatrix} \quad \text{for any } t,\ i,\ j,\ l.$$

We use the subgradient method [13] to get a lower bound for $v(P')$. The selection of the initial set of multipliers is crucial because the quality of the first solution depends very much on this choice. It should be noted that for an appropriate choice of multipliers λ_{jl}^t and μ_{il}^t, the solution of $LR(\lambda, \mu)$ must be close to a feasible solution. Otherwise, some of the constraints (1) or (3) would be violated and the corresponding term in the objective function would obtain worse values. For this reason, we propose the following set of initial multipliers:

1. Subproblem $LR2(\lambda, \mu)$.

$$\lambda_{jl}^t = \max_k b_{jkl}^t \quad \text{for all } j,\ l,\ t.$$

Once we know λ_{jl}^t, we describe the multipliers for $LR1(\lambda, \mu)$.

2. Subproblem $LR1(\lambda, \mu)$.

$$\mu_{il}^t = \max_j (c_{ijl}^t + \lambda_{jl}^t) d_{il}^t \quad \text{for all } i,\ l,\ t.$$

In addition, as Barros and Labbé suggest in [3], these results can be improved if the region of variation of the multipliers is reduced. Applying this technique to our problem, the corresponding feasible region for the multipliers is the following (see Appendix A for details on these results):

$$0 \leqslant \lambda_{jl}^t \leqslant \min_k \left\{ b_{jkl}^t + \frac{G_k^t}{\sum_{r \in \overline{T_{kt}}} C_k^r} \right\} \quad \forall j,\ \forall l,\ \forall t.$$

Then

$$\min_j \{(c_{ijl}^t + \lambda_{jl}^t) d_{il}^t\} \leqslant \mu_{il}^t \leqslant \min_j \left\{ \left(c_{ijl}^t + \frac{F_j^t}{\sum_{r \in \overline{T_{jt}}} W_j^r} + \lambda_{jl}^t \right) d_{il}^t \right\} \quad \forall i,\ \forall l,\ \forall t,$$

where, we denote by

$$\overline{T_{jt}} = \begin{cases} \{t, \ldots, T\} & \text{if } j \in J_0 \\ \{1, \ldots, t\} & \text{if } j \in J_c \end{cases} \quad \text{and} \quad \overline{T_{kt}} = \begin{cases} \{t, \ldots, T\} & \text{if } k \in K_0 \\ \{1, \ldots, t\} & \text{if } k \in K_c \end{cases}.$$

4. Heuristic to construct a feasible solution

In the previous section, we develop an ascent procedure to generate a good solution for the relaxed problem (DL). This solution is very often infeasible for our original problem (P). Therefore, we must develop an alternative procedure that starting from that solution constructs a good feasible solution for (P').

We propose the following scheme that consists of two different steps. The first step looks for capacities in each time period t. Both for plants and warehouses. Once these capacities have been established for meeting the demand, the second step looks for the best transportation plan between plants and warehouses, and between warehouses and customers. A detailed description of this procedure is given in the following paragraph.

Step 1. For each time period t compute the total capacity of all the open warehouses as well as the total demand in t. Let us denote by C_t the difference between the demand and the capacity in this time period.

Arrange in nonincreasing sequence with respect to C_t all those time periods where the capacity of the plants is not enough to cover the demand.

For every time period t_0 arranged according to the above process assign to all the warehouses j which are closed at t_0 the index

$$I(j,t_0) = \left[v(LR1_{jt_0}(\lambda,\mu)) - \overline{v}(LR1_j(\lambda,\mu))\right] \times \left[\max\left\{\frac{C_{t_0}}{W_j^{t_0}},1\right\}\right],$$

where

$$\overline{v}(LR1_j(\lambda,\mu)) := \begin{cases} v(LR1_{j1}(\lambda,\mu)) & \text{if } j \in J^1, \\ \Delta T(j) + v(LR1_j(\lambda,\mu)) & \text{if } j \in J^T, \\ v(LR1_j(\lambda,\mu)) & \text{otherwise.} \end{cases}$$

Remark. The reason for the above ordering is that the greater C_t, the larger the number of warehouses that have to be opened and this affects the remaining time periods. Then, $I(j,t_0)$ gives us a cost index of the cost which reflects the fact that warehouse j is opened at the time period t_0 rather than in the time period where it is currently open. To see this interpretation, just consider that $v(LR1_{jt_0}(\lambda,\mu)) - \overline{v}(LR1_j(\lambda,\mu))$ is the increment in the objective function if the warehouse j is opened at t_0 and $\max\left\{(C_{t_0}/W_j^{t_0}),1\right\}$ is the number of times that one should open the warehouse j to satisfy the uncovered demand.

The process consists of opening those warehouses in nondecreasing order of the index $I(\cdot,t_0)$ until the demand in that time period is fulfilled.

Once the process is finished, if there is excess capacity, one verifies whether there exists open warehouses whose capacity is less than or equal to the excess. If this happens, one should close those warehouses with greatest index among those verifying that their capacity is less than the excess of capacity of the whole process. This swapping process continues until all the open warehouses at t_0 have a capacity greater than or equal to C_{t_0}.

The same procedure has to be applied to the opening of plants. Obviously the capacity of the open plants in each time period has to be enough to satisfy the demand of the warehouses. The demand of the warehouses coincides with the demand of all the customers in a considered time period. The only difference in this step with respect to the previous one is that the index $I(k,t_0)$ is now computed based on $v(LR2_k(\lambda,\mu))$.

Step 2. Once the warehouses and plants open in each time period t_0 is known, we can replace the values of these binary variables in the formulation of (P'). Therefore, (P') is a continuous linear program that can be easily solved.

These two steps give us a feasible solution for (P'). In this process, we are intensively using the solution of our relaxed problems.

5. Computational study

The computational results presented in this section were designed to obtain the performance of our algorithm with respect to several test problems. The considered model combining dynamic aspects with multi-echelon location problems have not been considered previously in the literature. For this reason, no comparisons with other computational tests can be reported.

The computational study has been performed in a subcomplex (virtual machine) with six processors and $\frac{1}{2}$ Gb of RAM of a machine HP Exemplar SPP-1000 Series. The code has been written in C++ and uses subroutines of IMSL to solve linear programs. In addition, CPLEX 6.0 has been used to obtain exact solutions of medium sized problems by branch & bound using the parameters defined by default by the solver. The data have been generated randomly. The transportation cost c^t_{ijl} and b^t_{jkl} have been computed being proportional to the Euclidean distance among the location of final customers and warehouses, and plants and warehouses, respectively. The locations of all the facilities were uniformly distributed in the square $[1, 15] \times [1, 15]$. In addition, we assume that all these costs experience an increment between 10% and 25% in each time period (inflation rate, etc.)

The maintenance costs of warehouses and plants have been generated according to a uniform distribution $U(600, 1000)$. The demand follows a uniform distribution $U(0, 20)$. Finally, the capacity of warehouses and plants has been drawn uniformly in $U(50, 80)$.

The minimum number of plants and warehouses open at the first and the last time period depends on the difference between the total demand requested in each time period and the average of the capacity of warehouses (respectively plants) in that time period.

Table 1 describes the test problems that have been solved. Planning horizons from 1 to 4 periods have been considered. For each planning horizon we have solved seven different classes of problems. Each one of them differs by the data structure assigned to the parameters in their formulations. These problems are named "P1" to "P7". In this table, column "Customer" denotes the number of customers (I). The column "Warehouses" includes the number of warehouses distinguishing the number of warehouses that can be open in future time periods (J_0), the number of currently open warehouses (J_c) and $J = J_0 + J_c$. The column "Plants" includes the same information with respect to the number of plants. Finally, the column "Products" indicates the number of different commodities used in the problem.

Table 2 shows the size of each test problem for the considered planning horizons ($T = 1, T = 2, T = 3, T = 4$). The row "NV" describes the number of decision variables and the row "NC" the number of constraints for each time period in the problem (P').

Finally, Table 3 shows the results for the considered planning horizons ($T = 1, T = 2, T = 3, T = 4$) . For each planning horizon and problem class, at least 10 instances have been solved and the average results are reported. In this table, "H-Gap" denotes the percentage gap between the feasible solution obtained

Table 1
Description of test problems

	Customers	Warehouses			Plants			Products
	I	J_0	J_c	J	K_0	K_c	K	L
P1	10	4	1	5	4	1	5	2
P2	10	5	2	7	5	2	7	3
P3	20	7	3	10	7	3	10	2
P4	20	8	4	12	8	4	12	3
P5	30	10	5	15	10	5	15	2
P6	50	14	6	20	14	6	20	2
P7	75	25	15	40	25	15	40	2

Table 2
Size of test problems

		P1	P2	P3	P4	P5	P6	P7
$T = 1$	NV	160	371	620	1176	1380	2840	9280
	NC	52	81	102	146	152	222	394
$T = 2$	NV	320	742	1240	2352	2760	5680	18560
	NC	94	148	184	268	274	404	704
$T = 3$	NV	480	1113	1860	3528	4140	8520	27840
	NC	134	213	264	388	394	584	1014
$T = 4$	NV	640	1484	2480	4704	5520	11360	37120
	NC	174	278	344	508	514	764	1324

Table 3
Computational results

		P1	P2	P3	P4	P5	P6	P7
$T = 1$	H-Gap	0.4266	0.6012	0.8028	0.3358	0.9352	0.7442	0.9630
	Worst-H	1.8420	1.9861	2.0145	1.7342	2.1467	2.0688	2.4362
	CPU-H	17	17	22	35	35	68	2889
	N	820	442	437	494	428	462	906
	E-Gap	0.3903	0.1793	0.1986	0.3199	0.3138	* [a]	*
	Worst-E	1.9517	0.8965	0.8618	1.1391	0.9232	*	*
	CPU-E	0.12	0.36	0.97	3.3	2.53	*	*
$T = 2$	H-Gap	0.6132	0.4121	0.2468	1.0629	1.0563	1.0155	1.1598
	Worst-H	1.8234	1.5136	1.4762	2.2042	2.1968	2.4326	2.5143
	CPU-H	43	36	72	96	173	282	7650
	N	845	369	549	474	828	736	738
	E-Gap	0.1063	0.1897	0.2037	0.8160	0.9606	*	*
	Worst-E	1.0450	1.1040	1.4665	1.2432	3.1316	*	*
	CPU-E	0.28	1.29	36.2	25.41	63.71	*	*
$T = 3$	H-Gap	2.2613	2.2821	3.7425	2.6172	3.4055	2.6157	3.623
	Worst-H	3.6531	3.7133	4.8941	3.9502	4.5167	4.0101	4.6899
	CPU-H	101	131	115	369	216	811	9124
	N	786	820	389	863	387	748	472
	E-Gap	0.6045	2.2135	2.7941	1.0798	1.7346	*	*
	Worst-E	1.8435	3.3818	3.4343	1.4833	2.6560	*	*
	CPU-E	2.35	17.54	432.46	411.66	3988	*	*
$T = 4$	H-Gap	3.6978	4.3105	4.8207	2.3071	4.4896	2.2023	4.5081
	Worst-H	4.9105	5.9048	6.2153	3.8104	5.9109	3.6717	6.0984
	CPU-H	119	139	257	339	343	1097	14089
	N	739	425	570	371	425	489	390
	E-Gap	0.8474	1.3176	1.6553	1.0092	1.402	*	*
	Worst-E	2.7363	3.6528	2.7082	2.3836	2.5517	*	*
	CPU-E	6.94	80.67	1507	5138	6555	*	*

[a] * Means that no exact optimal solutions are available.

applying the heuristic and the greatest lower bound obtained in each instance between the continuous and the Lagrangean relaxation of (P'). "Worst-H" denotes the worst result used to compute the average H-Gap. "N" is the number of iterations needed by the heuristic algorithm and "CPU-H" is the average time in seconds used for these iterations. "E-Gap" denotes the percentage gap with respect to the exact solution of the problem obtained using CPLEX. This gap has been obtained with respect to 10 exact solutions in each case. "Worst-E" denotes the worst result used to compute the average E-Gap, and CPU-E is the average time in seconds used by CPLEX to solve the problems. Notice that the values of "E-Gap" are not complete. The reason for the missing values is that to obtain the exact solutions CPLEX solver needs prohibitive computational times. For instance, for problem P4 with $T = 4$ CPLEX took around 5000 seg. of CPU while for P5 with $T = 4$ CPLEX was even not able to obtain the exact solution in many cases. Summarizing the results shown in Table 3, the heuristic method that we propose to solve the multiperiod two-echelon multicommodity capacitated plant location problem provides solutions whose gaps (H-Gap) range between 0.24% and 5%. It is worth noting that these gaps are computed with respect to lower bounds of the optimal values. The variability in the H-Gap depends on the quality of the lower bound found in each case. In those cases where exact optimal solutions have been obtained the gaps (E-Gap) are much smaller and more stable. In these cases E-Gap ranges between 0.17 and 2.7. Finally, the CPU-E time is smaller than CPU-H for small-sized problems. However, when the size of the problem increases CPU-H is more stable and much smaller than CPU-E. These results show that our heuristic behaves acceptably well to solve this kind of problems.

6. Conclusions

The multiperiod two-echelon multicommodity capacitated plant location problem combines many features previously considered in the field of locational analysis which, as far as we know, have never been studied all together. Despite its difficulty, this is a natural model to formulate all those large-scale distribution models with seasonal demand.

In this paper, we propose a heuristic method to solve this problem. Our method is based on a Lagrangean relaxation which provides solutions (possibly infeasible for the original problem) but verifying the integrality constraints. In a second step, starting with these solutions we build feasible solutions of our original problem. We report computational results which show the gaps between the solutions that we propose and lower bounds of the optimal solution and exact solutions. The values of the gaps (H-Gap, E-Gap) and the computational times shown in Table 3 indicate that our heuristic is acceptable to solve the multiperiod two-echelon multicommodity capacitated plant location problem.

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Appendix A

In this section we prove Theorem 1 and we obtain a reduced feasible region for the multipliers of problem $LR(\lambda, \mu)$.

Proof of Theorem 1. Let (x,y,u,v) be a feasible solution of problem (P). We define (x',y',z,ζ) as:

$$x' = x; \quad y' = y;$$

$$\forall j \in J_0, \quad \begin{array}{l} z_j^1 = u_j^1 \\ z_j^t = u_j^t - u_j^{t-1} \quad \forall t > 1 \end{array} \qquad \text{and} \quad \forall j \in J_c, \quad \begin{array}{l} z_j^T = u_j^T \\ z_j^t = u_j^t - u_j^{t+1} \quad \forall t < T, \end{array}$$

$$\forall k \in K_0, \quad \begin{array}{l} \zeta_k^1 = v_k^1 \\ \zeta_k^t = v_k^t - v_k^{t-1} \quad \forall t > 1 \end{array} \qquad \text{and} \quad \forall k \in K_c, \quad \begin{array}{l} \zeta_k^T = v_k^T \\ \zeta_k^t = v_k^t - v_k^{t+1} \quad \forall t < T. \end{array}$$

By constraints (7) (respectively (8)) and (10) we obtain $z_j^t \in \{0,1\}\ \forall j,\ \forall t$ (respectively $\zeta_k^t \in \{0,1\}\ \forall k,\ \forall t$). In addition, by the definition of the variables z and ζ, we have:

$$u_j^t = \sum_{r \in T_{jt}} z_j^r \quad \text{and} \quad v_k^t = \sum_{r \in T_{kt}} \zeta_k^r.$$

Therefore, since (x,y,u,v) is a feasible solution of (P), it is straightforward to substitute u and v in the constraints of (P) to check that (x',y',z,ζ) verify the constraints of (P'). Hence, (x',y',z,ζ) is a feasible solution of (P').

On the other hand, by the definition of F_j^t and G_k^t we have:

$$\forall j \in J_0, \quad f_j^t = \begin{cases} F_j^t - F_j^{t+1} & \forall t < T, \\ F_j^T & \text{if } t = T. \end{cases} \quad \text{and} \quad \forall j \in J_c, \quad f_j^t = \begin{cases} F_j^t - F_j^{t-1} & \forall t > 1, \\ F_j^1 & \text{if } t = 1. \end{cases}$$

$$\forall k \in K_0, \quad g_k^t = \begin{cases} G_k^t - G_k^{t+1} & \forall t < T, \\ G_k^T & \text{if } t = T. \end{cases} \quad \text{and} \quad \forall k \in K_c, \quad g_k^t = \begin{cases} G_k^t - G_k^{t-1} & \forall t > 1, \\ G_k^1 & \text{if } t = 1. \end{cases}$$

Then we obtain

$$\begin{aligned} \sum_{t=1}^{T} \sum_{j=1}^{m} f_j^t u_j^t &= \sum_{j \in J_0} \left[\sum_{t=1}^{T-1} (F_j^t - F_j^{t+1}) \sum_{r=1}^{t} z_j^r + F_j^T \sum_{r=1}^{T} z_j^r \right] + \sum_{j \in J_c} \left[F_j^1 \sum_{r=1}^{T} z_j^r + \sum_{t=2}^{T} (F_j^t - F_j^{t-1}) \sum_{r=t}^{T} z_j^r \right] \\ &= \sum_{j \in J_0} \left[\sum_{t=1}^{T} F_j^t z_j^t \right] + \sum_{j \in J_c} \left[\sum_{t=1}^{T} F_j^t z_j^t \right] = \sum_{t=1}^{T} \sum_{j=1}^{m} F_j^t z_j^t, \end{aligned} \tag{A.1}$$

and in the same way,

$$\sum_{t=1}^{T} \sum_{k=1}^{p} g_k^t v_k^t = \sum_{t=1}^{T} \sum_{k=1}^{p} G_k^t \zeta_k^t. \tag{A.2}$$

Therefore, $g(x,y,u,v) = f(x',y',z,\zeta)$.

Conversely, let (x',y',z,ζ) be a feasible solution of problem (P'). We define (x,y,u,v) as:

$$x = x', \quad y = y', \quad u_j^t = \sum_{r \in T_{jt}} z_j^r \quad \forall j,t, \quad v_k^t = \sum_{r \in T_{kt}} \zeta_k^r \quad \forall j,t.$$

By constraints (7a) (respectively (8a)) and (10a) we obtain $u_j^t \in \{0,1\}\ \forall j,\ \forall t$ (respectively $v_k^t \in \{0,1\}\ \forall k, \forall t$). In addition, by the definition of the variables u and constraints (7a) we obtain

$u_j^1 = \sum_{t=1}^{T} z_j^t = 1 \ \forall j \in J_c$ and by (10a) we obtain $u_j^t \geqslant u_j^{t+1} \ \forall j \in J_c \ \forall t$ and $u_j^t \leqslant u_j^{t+1} \ \forall j \in J_0 \ \forall t$. Then, the variables u fulfill the constraints (7). In the same way, by (8a) and (10a) one can prove that the variables v verify the constraints (8). Substituting z and ζ by u and v in the remainder constraints of (P') it is straightforward that u and v fulfill the constraints of (P). Therefore, (x, y, u, v) is a feasible solution of (P) and by (A.1) and (A.2) we have, $f(x', y', z, \zeta) = g(x, y, u, v)$.

Reduced feasible region for the multipliers of problem $LR(\lambda, \mu)$.

Let (LP) denote the continuous relaxation of Problem (P') and (DLP) the continuous dual of (LP). Then, the mathematical formulation of problem (DLP) is

$$(DLP) \quad \max \sum_{t=1}^{T} \sum_{i=1}^{n} \sum_{l=1}^{q} \mu_{il}^t + \sum_{j \in J_c} \gamma_j - \sum_{j \in J_0} \gamma_j + ND^1 \phi^1 + ND^T \phi^T + \sum_{k \in K_c} \delta_k - \sum_{k \in K_0} \delta_k + NC^1 \psi^1 + NC^T \psi^T$$

s.t.

$$\mu_{il}^t - d_{il}^t \alpha_j^t - d_{il}^t \lambda_{jl}^t \leqslant c_{ijl}^t d_{il}^t \quad \forall i, j, l, t, \tag{A.3}$$

$$W_j^t \lambda_{jl}^t - W_j^t \beta_k^t \leqslant W_j^t b_{jkl}^t \quad \forall j, k, l, t, \tag{A.4}$$

$$\left(\sum_{r=1}^{T} W_j^r \right) \alpha_j^1 - \gamma_j + \phi^T + \phi^1 \leqslant F_j^1 \quad \forall j \in J_0, \tag{A.5}$$

$$\left(\sum_{r=t}^{T} W_j^r \right) \alpha_j^t - \gamma_j + \phi^T \leqslant F_j^t \quad \forall j \in J_0, \ t = 2, \ldots, T, \tag{A.6}$$

$$\left(\sum_{r=1}^{T} W_j^r \right) \alpha_j^T + \gamma_j + \phi^1 + \phi^T \leqslant F_j^T \quad \forall j \in J_c, \tag{A.7}$$

$$\left(\sum_{r=1}^{t} W_j^r \right) \alpha_j^t - \gamma_j + \phi^1 \leqslant F_j^t \quad \forall j \in J_c, \ t = 1, \ldots, T-1, \tag{A.8}$$

$$\left(\sum_{r=1}^{T} C_k^r \right) \beta_k^1 - \delta_k + \psi^T + \psi^1 \leqslant G_k^1 \quad \forall k \in K_0, \tag{A.9}$$

$$\left(\sum_{r=t}^{T} C_k^r \right) \beta_k^t - \delta_k + \psi^T \leqslant G_k^t \quad \forall k \in K_0, \ t = 2, \ldots, T, \tag{A.10}$$

$$\left(\sum_{r=1}^{T} C_k^r \right) \beta_k^T + \delta_k + \psi^1 + \psi^T \leqslant G_k^T \quad \forall k \in K_c, \tag{A.11}$$

$$\left(\sum_{r=1}^{t} C_k^r \right) \beta_k^t - \delta_k + \psi^1 \leqslant G_k^t \quad \forall k \in K_c, \ t = 1, \ldots, T-1, \tag{A.12}$$

$$\gamma_j \text{ unrestricted } \forall j \in J_c: \quad \gamma_j \geqslant 0 \quad \forall j \in J_0, \tag{A.13}$$

$$\delta_k \text{ unrestricted } \forall k \in K_c: \quad \delta_k \geqslant 0 \quad \forall k \in K_0, \tag{A.14}$$

$$\mu_{il}^t, \alpha_j^t, \lambda_{jl}^t, \beta_k^t \geqslant 0 \quad \forall i, \ \forall j, \ \forall k, \ \forall l, \ \forall t, \tag{A.15}$$

$$\phi^1, \phi^T, \psi^1, \psi^T \geqslant 0. \tag{A.16}$$

The feasible region of (*DLP*) is used to obtain bounds on the range of variation of multipliers (λ, μ).

Lemma 1. *A reduced feasible region for the multipliers* (λ, μ) *is*

$$0 \leqslant \lambda^t_{jl} \leqslant \min_k \left\{ b^t_{jkl} + \frac{G^t_k}{\sum_{r \in \overline{T}_{kt}} C^r_k} \right\} \quad \forall j,\ \forall l,\ \forall t.$$

$$\min_j \{(c^t_{ijl} + \lambda^t_{jl}) d^t_{il}\} \leqslant \mu^t_{il} \leqslant \min_j \left\{ \left(c^t_{ijl} + \frac{F^t_j}{\sum_{r \in \overline{T}_{kt}} W^r_j} + \lambda^t_{jl} \right) d^t_{il} \right\} \quad \forall i,\ \forall l,\ \forall t.$$

Proof. The constraints of (*DLP*) allow us to establish a reduced feasible region for the dual variables:

$$\mu^t_{il} \leqslant (c^t_{ijl} + \alpha^t_j + \lambda^t_{jl}) d^t_{il} \quad \forall i,\ \forall j,\ \forall l,\ \forall t, \tag{A.17}$$

$$\lambda^t_{jl} \leqslant b^t_{jkl} + \beta^t_k \quad \forall j,\ \forall k,\ \forall l,\ \forall t. \tag{A.18}$$

In addition, it holds

$$\text{if } j \in J_0 \quad \Rightarrow \quad \alpha^t_j \leqslant \frac{F^t_j}{\sum_{r=t}^T W^r_j} + \frac{\gamma_j}{\sum_{r=t}^T W^r_j} \quad \text{with } \gamma_j \geqslant 0,$$

$$\text{if } j \in J_c \quad \Rightarrow \quad \begin{cases} \alpha^t_j \leqslant \frac{F^t_j}{\sum_{r=1}^t W^r_j} - \frac{\gamma_j}{\sum_{r=1}^t W^r_j} & \text{if } \gamma_j \leqslant 0 \\ \alpha^t_j \leqslant \frac{F^t_j}{\sum_{r=1}^t W^r_j} & \text{if } \gamma_j \geqslant 0, \end{cases}$$

$$\text{if } k \in K_0 \quad \Rightarrow \quad \beta^t_k \leqslant \frac{G^t_k}{\sum_{r=t}^T C^r_k} + \frac{\delta_k}{\sum_{r=t}^T C^r_k} \quad \text{with } \delta_k \geqslant 0,$$

$$\text{if } k \in K_c \quad \Rightarrow \quad \begin{cases} \beta^t_k \leqslant \frac{G^t_k}{\sum_{r=1}^t C^r_k} - \frac{\delta_k}{\sum_{r=1}^t C^r_k} & \text{if } \delta_k \leqslant 0 \\ \beta^t_k \leqslant \frac{G^t_k}{\sum_{r=1}^t C^r_k} & \text{if } \delta_k \geqslant 0. \end{cases}$$

Therefore, if we assume $j \in J_0$ and $k \in K_0$ with $\delta_k \leqslant 0$, we have

$$\mu^t_{il} \leqslant \left(c^t_{ijl} + \frac{F^t_j}{\sum_{r=t}^T W^r_j} + \frac{\gamma_j}{\sum_{r=t}^T W^r_j} + b^t_{jkl} + \frac{G^t_k}{\sum_{r=1}^t C^r_k} - \frac{\delta_k}{\sum_{r=1}^t C^r_k} \right) d^t_{il}.$$

Then, if μ^t_{il} achieves its maximum value, the following two terms appear in the objective function of (*DLP*):

$$\left(\frac{d^t_{il}}{\sum_{r=t}^T W^r_j} - 1 \right) \gamma_j \quad \text{and} \quad \left(1 - \frac{d^t_{il}}{\sum_{r=1}^t C^r_k} \right) \delta_k \quad \text{with } \gamma_j \geqslant 0,\ \ \delta_k \leqslant 0.$$

Now, since we have that

$$\frac{d^t_{il}}{\sum_{r=t}^T W^r_j} < 1, \quad \text{and} \quad \frac{d^t_{il}}{\sum_{r=1}^t C^r_k} < 1,$$

we can conclude that $\gamma_j = 0$ and $\delta_k = 0$. Analogously, if we assume that either $j \in J_0$ and $k \in K_0$ or $j \in J_c$ and $k \in K_0$ or $k \in K_c$ we obtain the same conclusion.

From (A.17) and (A.18) we obtain

$$\mu_{il}^t \leqslant \min_j \left\{ \left(c_{ijl}^t + \frac{F_j^t}{\sum_{r\in\overline{T}_{jt}} W_j^r} + \lambda_{jl}^t \right) d_{il}^t \right\} \quad \forall i,\ \forall l,\ \forall t,$$

$$\lambda_{jl}^t \leqslant \min_k \left\{ b_{jkl}^t + \frac{G_k^t}{\sum_{r\in\overline{T}_{kt}} C_k^r} \right\} \quad \forall j,\ \forall l,\ \forall t.$$

On the other hand, since (DLP) is a maximization problem we can use (A.3) to have

$$\mu_{il}^t = (c_{ijl}^t + \alpha_j^t + \lambda_{jl}^t)d_{il}^t \geqslant (c_{ijl}^t + \lambda_{jl}^t)d_{il}^t \geqslant \min_j\{(c_{ijl}^t + \lambda_{jl}^t)d_{il}^t\} \quad \forall i,\ \forall l,\ \forall t.$$

In conclusion, a reduced feasible region of multipliers can be given as follows:

$$0 \leqslant \lambda_{jl}^t \leqslant \min_{\mathrm{k}} \left\{ b_{jkl}^t + \frac{G_k^t}{\sum_{r\in\overline{T}_{kt}} C_k^r} \right\} \quad \forall j,\ \forall l,\ \forall t.$$

Then, once the value of λ_{jl}^t is known we get for μ_{il}^t the following region:

$$\min_j\{(c_{ijl}^t + \lambda_{jl}^t)d_{il}^t\} \leqslant \mu_{il}^t \leqslant \min_j \left\{ \left(c_{ijl}^t + \frac{F_j^t}{\sum_{r\in\overline{T}_{jt}} W_j^r} + \lambda_{jl}^t \right) d_{il}^t \right\} \quad \forall i,\ \forall l,\ \forall t.$$

References

[1] C.H. Aikens, Facility location models for distribution planning, European Journal of Operational Research 22 (1985) 263–279.

[2] J. Barceló, E. Fernández, K. Jornsten, Computational results from a new lagrangean relaxation algorithm for the capacitated plant location problem, European Journal of Operational Research 53 (1991) 38–45.

[3] A.I. Barros, M. Labbé, A general model for the uncapacitated facility and depot location problem, Location Science 2 (1994) 173–191.

[4] J.E. Beasley, Lagrangean heuristics for location problems, European Journal of Operational Research 65 (1993) 383–399.

[5] T.G. Crainic, L. Delorme, Dual-ascent procedures for multicommodity location-allocation problems with balancing requirements, Transportation Science 27 (2) (1993) 90–101.

[6] P. Chardaire, A. Sutter, M.C. Costa, Solving the dynamic facility location problem, Networks 28 (1996) 117–124.

[7] M. Daskin, Network and Discrete Location, Wiley, New York, 1995.

[8] Z. Drezner, Facility Location, Springer, Berlin, 1995.

[9] D. Erlenkotter, A dual-based procedure for uncapacitated facility location, Operational Research 26 (6) (1978) 992–1009.

[10] M. Fisher, The Lagrangean relaxation method for solving integer programming problems, Management Science 27 (1) (1981) 1–18.

[11] M. Guignard, M. Rosenwein, A application-oriented guide for designing Lagrangean dual ascent algorithms, European Journal of Operational Research 43 (1989) 197–205.

[12] M. Guignard, K. Opaswongkarn, Lagrangean dual ascent algorithms for computing bounds in capacitated plant location problems, European Journal of Operational Research 46 (1990) 73–83.

[13] M. Held, P. Wolfe, H. Crowder, Validation of subgradient optimization, Mathematical Programming 6 (1974) 62–88.

[14] K. Holmberg, K. Jornsten, Dual search procedures for the exact formulation of the simple plant location problem with spatial interaction, Location Science 4 (1996) 83–100.

[15] L. Kaufman, M.V. Eede, P. Hansen, A plant and warehouse location problem, Operational Research Quarterly 28 (1977) 547–554.

[16] J. Krarup, P.M. Pruzan, The simple plant location problem: Survey and synthesis, European Journal of Operational Research 12 (1983) 36–81.

[17] A. Marín, Análisis y resolución de problemas de localización discreta en dos etapas mediante técnicas basadas en la descomposición lagrangiana, Thesis, Dpto. Matemática Aplicada y Estadística, Universidad de Murcia, 1996.
[18] H. Pirkul, V. Jayaraman, Production transportation and distribution planning in a multicommodity tri-echelon system, Transportation Science 30 (4) (1996) 291–302.
[19] G.M. Roodman, L.B. Schwartz, Optimal and heuristic facility phase-out strategies, AIIE Transactions 7 (1975) 177–184.
[20] D. Tcha, B. Lee, A branch-and-bound algorithm for the multi-level uncapacitated facility location problem, European Journal of Operational Research 18 (1984) 35–43.
[21] T.J. van Roy, D. Erlenkotter, A dual-based procedure for dynamic facility location, Management Science 28 (10) (1982) 1091–1105.
[22] A. Warszawski, Multi-dimensional location problems, Operational Research Quarterly 24 (1973) 165–179.